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Periodic and discrete Zak bases

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Abstract

Weyl's unitary operators for displacement in position and momentum commute with one another if the product of the elementary displacements equals Planck's constant. Then, their common eigenstates constitute the Zak basis, with each state specified by two phase parameters. Accordingly, the transformation function from the position basis to the Zak basis maps the Hilbert space on the line onto the Hilbert space on the torus. This mapping is one to one provided that the Zak basis states are periodic functions of their phase parameters, but then the mapping cannot be continuous on the whole torus. With the periodicity of the Zak basis enforced, the basis has a double Fourier series. The Fourier coefficients identify a discrete basis which complements the periodic Zak basis to form a pair of mutually unbiased bases. The discrete basis states are the common eigenstates of the two complementary partners to the two unitary displacement operators. These partner operators are of angular-momentum type, with integer eigenvalues, and generate the fundamental rotations of the torus. Conversely, the displacement operators are the ladder operators for their partners. For each consistent phase convention for the periodic Zak basis, and thus for the line-onto-torus mapping, there is a corresponding discrete Zak basis and a corresponding pair of partner operators. Examples of particular interest are the conventions that give a continuous mapping in one phase parameter or are symmetric in both phase parameters. The latter emphasizes the Heisenberg–Weyl symmetry between position and momentum. We discuss briefly the relation between the Zak bases and Aharonov's modular operators. Finally, as an application of the Zak operators for the torus, we mention how they can be used to associate with the single degree of freedom of the line a pair of genuine qubits that are potentially entangled.

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1. Introduction

A convenient choice of variables may render simple a complex physical problem. Zak introduced [1, 2] what he termed 'the kq representation' as a particularly convenient mode of description for electrons in solids [3, 4]. Zak's work was based on Weyl's [5] presentation of unitary displacement operators for coordinates and momenta (obtained by exponentiating both the momentum and the coordinate operators) and noting that for some particular choice of parameters these displacement operators commute. Weyl's work [5] led to several studies that dealt with both mathematical and physical representations wherein the spatial coordinates are defined modulo some conveniently selected length a and the momenta modulo $2\pi/a$ (in units for which $\hbar = 1$). However, it was Zak [1, 2] who was the first to recognize the wide context of the approach and to study it in a systematic way [6].

The Zak transform of a state may be regarded as a mixed position–momentum (or time–frequency) representation of the state. The transform enabled Zak [3, 4] to reduce the problem of an electron in a periodic potential in a constant magnetic field to a form that made possible studies of further physical applications—to digital data transmission, for instance, as is considered in [7].

Reference [6] is a thorough review of the Zak transform and its applications up to the year 1988; for recent applications, see [8–10] for example. In the more recent literature, the representation finds its widest use in signal processing where time plays the role of the spatial coordinate and frequency that of the momentum; see [7, 11, 12] and references therein.

In the present work, which we regard as a contribution to our understanding of quantum kinematics, we consider the Zak transform as a mapping of the Hilbert space on a line to the Hilbert space on a torus. This allows us to study, in section 2, periodic and discrete Zak bases which are mutually unbiased [13, 14]. In section 3, we use some freedom in the permissible definition of the phase of the periodic wavefunctions in the Zak basis to present and discuss three convenient phase choices. Section 4 is devoted to the study of the discrete Zak basis; we display and discuss the Wigner functions [15] (adopting the conventions of [16]; see the footnote at (4.5)) that are associated with the wavefunctions in these bases. In section 5, the position and momentum operators are given as differential operators in the Zak representation of section 3. Section 6 deals with the relation of the Zak bases and operators to the modular variables introduced by Aharonov *et al* in [17]. A possible application for the Zak bases in quantum information theory is hinted at in section 7. We close with a summary.

2. Setting the stage

We consider a single continuous quantum degree of freedom, the motion along the x axis, for which the position operator X and the momentum operator P constitute the fundamental complementary pair of Hermitian dynamical variables. They obey the Heisenberg–Born commutation relation $[X, P] = \mathrm{i}\hbar$, and we adopt the usual normalization conventions for their complete sets of eigenbras and eigenkets.

We denote Weyl's unitary operator for the momentum shift $P \to P - p_0$ by U and that for the position shift $X \to X - x_0$ by V,

$$U = \mathrm{e}^{-\mathrm{i}p_0 X/\hbar}, \qquad V = \mathrm{e}^{\mathrm{i}x_0 P/\hbar}. \tag{2.1}$$

When both are acting on an operator, a function of X and P, their order is irrelevant,

$$\begin{aligned} f(X, P) \to f(X - x_0, P - p_0) &= U^\dagger V^\dagger f(X, P) U V \\ &= V^\dagger U^\dagger f(X, P) V U. \end{aligned} \tag{2.2}$$

But the order does matter for the transformation of bras and kets, as illustrated by

$$\begin{aligned}\langle x|UV &= \mathrm{e}^{-\mathrm{i}p_0x/\hbar}\langle x|V = \mathrm{e}^{-\mathrm{i}p_0x/\hbar}\langle x+x_0|,\\ \langle x|VU &= \langle x+x_0|U = \mathrm{e}^{-\mathrm{i}p_0(x+x_0)/\hbar}\langle x+x_0|,\end{aligned} \tag{2.3}$$

unless the product p_0x_0 is an integer multiple of Planck's constant $h = 2\pi\hbar$. Following Zak [1, 2], we therefore choose the two displacements such that

$$p_0x_0 = 2\pi\hbar, \qquad UV = VU. \tag{2.4}$$

The product p_0x_0 then equals the phase space area that is roughly associated with one quantum state, and it is fitting to regard p_0 and x_0 as corresponding atomic units for momentum and position. This is emphasized, for example, by the relations

$$\begin{aligned}&U = \mathrm{e}^{-2\pi\mathrm{i}X/x_0}, \qquad V = \mathrm{e}^{2\pi\mathrm{i}P/p_0},\\ &2\pi\mathrm{i}[P/p_0, X/x_0] = 1.\end{aligned} \tag{2.5}$$

Now that U and V commute, they have common eigenstates, which we label by the real phases α and β that are associated with the unit-modulus eigenvalues,

$$U|\alpha,\beta\rangle = |\alpha,\beta\rangle\,\mathrm{e}^{-\mathrm{i}\alpha}, \qquad V|\alpha,\beta\rangle = |\alpha,\beta\rangle\,\mathrm{e}^{\mathrm{i}\beta}. \tag{2.6}$$

Since the characterizing eigenvalues are 2π periodic in α and β, we require that the eigenstates are periodic as well,

$$|\alpha,\beta\rangle = |\alpha+2\pi,\beta\rangle = |\alpha,\beta+2\pi\rangle. \tag{2.7}$$

In the completeness relations, then

$$\int_{(2\pi)}\frac{\mathrm{d}\alpha}{2\pi}\int_{(2\pi)}\frac{\mathrm{d}\beta}{2\pi}|\alpha,\beta\rangle\langle\alpha,\beta| = 1, \tag{2.8}$$

the integration is over any 2π interval, and the wavefunctions of the Zak representation, $\psi(\alpha,\beta) = \langle\alpha,\beta|\rangle$, are functions on the torus, not on the square.

In the orthonormality relation

$$\langle\alpha,\beta|\alpha',\beta'\rangle = (2\pi)^2\delta^{(2\pi)}(\alpha-\alpha')\delta^{(2\pi)}(\beta-\beta'), \tag{2.9}$$

we meet the periodic delta function

$$\delta^{(2\pi)}(\varphi) = \sum_{k=-\infty}^{\infty}\delta(\varphi-2\pi k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k\varphi}. \tag{2.10}$$

It appears also in the explicit construction of the projector on an α,β state in terms of the unitary shift operators,

$$\begin{aligned}|\alpha,\beta\rangle\langle\alpha,\beta| &= \sum_{j,k=-\infty}^{\infty}(\mathrm{e}^{\mathrm{i}\alpha}U)^j(\mathrm{e}^{-\mathrm{i}\beta}V)^k\\ &= (2\pi)^2\delta^{(2\pi)}(p_0X/\hbar-\alpha)\delta^{(2\pi)}(x_0P/\hbar-\beta).\end{aligned} \tag{2.11}$$

The Fourier coefficient kets $|l,m\rangle$ that are defined by the double Fourier series of $|\alpha,\beta\rangle$,

$$|\alpha,\beta\rangle = \sum_{l,m=-\infty}^{\infty}|l,m\rangle\,\mathrm{e}^{-\mathrm{i}(l\alpha-m\beta)}, \tag{2.12}$$

form the discrete Zak basis that is normalized in accordance with

$$\langle l,m|l',m'\rangle = \delta_{mm'}\delta_{ll'}, \qquad \sum_{l,m=-\infty}^{\infty}|l,m\rangle\langle l,m| = 1. \tag{2.13}$$

Upon observing that

$$\langle\alpha,\beta|l,m\rangle = \mathrm{e}^{\mathrm{i}(l\alpha - m\beta)}, \qquad |\langle\alpha,\beta|l,m\rangle| = 1, \tag{2.14}$$

we note that the periodic and the discrete Zak bases are mutually unbiased.

We regard the discrete basis states $|l,m\rangle$ as common eigenstates of two integer operators,

$$L|l,m\rangle = |l,m\rangle l, \qquad M|l,m\rangle = |l,m\rangle m. \tag{2.15}$$

Their action on the continuous Zak states amounts to differentiation,

$$\langle\alpha,\beta|L = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\alpha}\langle\alpha,\beta|, \qquad \langle\alpha,\beta|M = \mathrm{i}\frac{\partial}{\partial\beta}\langle\alpha,\beta|, \tag{2.16}$$

which implies that they are the generators of the fundamental rotations of the torus,

$$\langle\alpha,\beta|\,\mathrm{e}^{\mathrm{i}(\alpha' L - \beta' M)} = \langle\alpha+\alpha',\beta+\beta'|. \tag{2.17}$$

Conversely, the unitary displacement operators are ladder operators for the discrete Zak basis,

$$U^{l'}V^{m'}|l,m\rangle = |l-l',m-m'\rangle. \tag{2.18}$$

The commutation relations

$$\begin{aligned} &[U,V]=0, && [U,L]=U, && [U,M]=0, \\ &[V,M]=V, && [M,L]=0, && [V,L]=0 \end{aligned} \tag{2.19}$$

reveal that the pairs (U,L) and (V,M) are the dynamical variables of two *independent* degrees of freedom of azimuth–angular-momentum type. These are, of course, the two degrees of freedom of the torus.

3. Wavefunctions: periodic Zak basis

As a consequence of the eigenket equations (2.6), the position wavefunction $\langle x|\alpha,\beta\rangle$ of the periodic Zak state, which is the transformation function between the position representation and the periodic Zak representation, obeys the functional equations

$$\begin{aligned} \langle x|\alpha,\beta\rangle &= \mathrm{e}^{\mathrm{i}(\alpha - p_0 x/\hbar)}\langle x|\alpha,\beta\rangle \\ &= \mathrm{e}^{-\mathrm{i}\beta}\langle x+x_0|\alpha,\beta\rangle. \end{aligned} \tag{3.1}$$

The corresponding equations for the momentum wavefunction are

$$\begin{aligned} \langle p|\alpha,\beta\rangle &= \mathrm{e}^{\mathrm{i}\alpha}\langle p+p_0|\alpha,\beta\rangle \\ &= \mathrm{e}^{-\mathrm{i}(\beta - x_0 p/\hbar)}\langle p|\alpha,\beta\rangle. \end{aligned} \tag{3.2}$$

Their general solution is

$$\begin{aligned} \langle x|\alpha,\beta\rangle &= \frac{2\pi}{\sqrt{x_0}}\chi(\alpha,\beta)\exp\left(-\mathrm{i}\frac{\alpha\beta}{4\pi}\right)\mathrm{e}^{\mathrm{i}\beta x/x_0}\delta^{(2\pi)}(p_0x/\hbar-\alpha), \\ \langle p|\alpha,\beta\rangle &= \frac{2\pi}{\sqrt{p_0}}\chi(\alpha,\beta)\exp\left(\mathrm{i}\frac{\alpha\beta}{4\pi}\right)\mathrm{e}^{-\mathrm{i}\alpha p/p_0}\delta^{(2\pi)}(x_0p/\hbar-\beta), \end{aligned} \tag{3.3}$$

where $\chi(\alpha,\beta)$ must be of unit modulus,

$$|\chi(\alpha,\beta)| = 1, \tag{3.4}$$

for consistency with the normalization (2.8), and the periodicity of (2.7) requires that $\chi(\alpha,\beta)$ obeys

$$\chi(\alpha,\beta) = \chi(\alpha+2\pi,\beta)\,\mathrm{e}^{-\mathrm{i}\beta/2} = \mathrm{e}^{\mathrm{i}\alpha/2}\chi(\alpha,\beta+2\pi). \tag{3.5}$$

Any $\chi(\alpha, \beta)$ permitted by these constraints is an acceptable phase convention for the periodic Zak bases.

Equation (3.5) implies that

$$\chi(\alpha, \beta) = (-1)^{ab}\,\mathrm{e}^{\mathrm{i}(a\beta - \alpha b)/2}\chi(\alpha - 2\pi a, \beta - 2\pi b) \tag{3.6}$$

if a, b are integers, so that $\chi(\alpha, \beta)$ is specified by stating its values within a standard $2\pi \times 2\pi$ square. We find the following three choices for $\chi(\alpha, \beta)$ particularly natural and interesting:

$$\begin{aligned}
&\text{(a)}\quad \chi(\alpha, \beta) = \exp\left(\mathrm{i}\alpha\left(\frac{\beta}{4\pi} - \left\lfloor\frac{\beta}{2\pi}\right\rceil\right)\right),\\
&\text{(b)}\quad \chi(\alpha, \beta) = \exp\left(-\mathrm{i}\beta\left(\frac{\alpha}{4\pi} - \left\lfloor\frac{\alpha}{2\pi}\right\rceil\right)\right),\\
&\text{(c)}\quad \chi(\alpha, \beta) = (-1)^{\lfloor\frac{\alpha}{2\pi}\rceil\lfloor\frac{\beta}{2\pi}\rceil}\exp\left(\mathrm{i}\left(\left\lfloor\frac{\alpha}{2\pi}\right\rceil\frac{\beta}{2} - \frac{\alpha}{2}\left\lfloor\frac{\beta}{2\pi}\right\rceil\right)\right),
\end{aligned} \tag{3.7}$$

where

$$\lfloor z\rceil = \int_0^z \mathrm{d}\zeta \sum_{k=-\infty}^{\infty}\delta\left(\zeta - k - \frac{1}{2}\right) = z - \frac{1}{\pi}\arctan\big(\tan(\pi z)\big) \tag{3.8}$$

denotes the integer closest to z. $\chi(\alpha, \beta)$ of choice (a) is continuous in α but discontinuous in β, that of choice (b) is discontinuous in α and continuous in β, and that resulting from choice (c) is symmetric in α and β in the sense of $\chi(\alpha, \beta) = \chi(-\beta, \alpha) = \chi(\beta, \alpha)^*$ and discontinuous in both.

As these examples demonstrate, discontinuities are a generic feature of the wavefunctions we are dealing with here, and of other functions derived from them. It will be necessary on various occasions to assign values to such functions at their discontinuities. We employ the convention to assign half the sum of the one-sided limits, for which

$$\begin{aligned}
&\mathrm{sgn}(x) = \begin{cases} +1 & \text{for } x > 0,\\ 0 & \text{for } x = 0,\\ -1 & \text{for } x < 0,\end{cases}\\
&[\mathrm{sgn}(x)]^2 = 1 \quad \text{for all } x,\\
&\mathrm{e}^{\mathrm{i}\varphi\,\mathrm{sgn}(x)} = \begin{cases} \mathrm{e}^{\mathrm{i}\varphi} & \text{for } x > 0,\\ \cos\varphi & \text{for } x = 0,\\ \mathrm{e}^{-\mathrm{i}\varphi} & \text{for } x < 0\end{cases}
\end{aligned} \tag{3.9}$$

are illustrative examples. As an application, and for future reference, we note that

$$\begin{aligned}
&\chi(\alpha, \beta)^*\frac{1}{\mathrm{i}}\frac{\partial}{\partial\alpha}\chi(\alpha, \beta) = \begin{cases}
\text{(a)} & \dfrac{\beta}{4\pi} - \left\lfloor\dfrac{\beta}{2\pi}\right\rceil,\\
\text{(b)} & -\dfrac{\beta}{4\pi} + \delta^{(2\pi)}(\alpha - \pi)\sin\beta,\\
\text{(c)} & -\dfrac{1}{2}\left\lfloor\dfrac{\beta}{2\pi}\right\rceil + \delta^{(2\pi)}(\alpha - \pi)(-1)^{\lfloor\frac{\beta}{2\pi}\rceil}\sin\dfrac{\beta}{2},
\end{cases}\\
&\chi(\alpha, \beta)^*\mathrm{i}\frac{\partial}{\partial\beta}\chi(\alpha, \beta) = \begin{cases}
\text{(a)} & -\dfrac{\alpha}{4\pi} + \delta^{(2\pi)}(\beta - \pi)\sin\alpha,\\
\text{(b)} & \dfrac{\alpha}{4\pi} - \left\lfloor\dfrac{\alpha}{2\pi}\right\rceil,\\
\text{(c)} & -\dfrac{1}{2}\left\lfloor\dfrac{\alpha}{2\pi}\right\rceil + \delta^{(2\pi)}(\beta - \pi)(-1)^{\lfloor\frac{\alpha}{2\pi}\rceil}\sin\dfrac{\alpha}{2}
\end{cases}
\end{aligned} \tag{3.10}$$

for the respective choices in (3.7).

Further, as an important consequence of (3.5), we have the two Fourier series

$$\chi(\alpha,\beta) = \exp\left(-\mathrm{i}\frac{\alpha\beta}{4\pi}\right)\sum_{m=-\infty}^{\infty}\lambda(\alpha-2\pi m)\,\mathrm{e}^{\mathrm{i}m\beta}$$
$$= \exp\left(\mathrm{i}\frac{\alpha\beta}{4\pi}\right)\sum_{l=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}l\alpha}\mu(\beta-2\pi l), \qquad (3.11)$$

with

$$\lambda(\alpha) = \int_{(2\pi)}\frac{\mathrm{d}\beta}{2\pi}\exp\left(\mathrm{i}\frac{\alpha\beta}{4\pi}\right)\chi(\alpha,\beta),$$
$$\mu(\beta) = \int_{(2\pi)}\frac{\mathrm{d}\alpha}{2\pi}\exp\left(-\mathrm{i}\frac{\alpha\beta}{4\pi}\right)\chi(\alpha,\beta), \qquad (3.12)$$

where the integrands are 2π periodic functions of the respective integration variable. Either one of the single-argument functions $\lambda(\alpha)$ or $\mu(\beta)$ introduced here, which are related to each other by Fourier transformation,

$$\lambda(\alpha) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\beta}{2\pi}\exp\left(\mathrm{i}\frac{\alpha\beta}{2\pi}\right)\mu(\beta), \qquad \mu(\beta) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}\exp\left(-\mathrm{i}\frac{\alpha\beta}{2\pi}\right)\lambda(\alpha), \qquad (3.13)$$

can thus be used for a complete specification of $\chi(\alpha,\beta)$.

The restriction imposed by the normalization (3.4) can be stated as

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi}[\mathrm{e}^{\mathrm{i}l\alpha}\lambda(\alpha-2\pi m)]^*[\mathrm{e}^{\mathrm{i}l'\alpha}\lambda(\alpha-2\pi m')] = \delta_{ll'}\delta_{mm'} \qquad (3.14)$$

or equivalently as

$$\sum_{l,m=-\infty}^{\infty}[\mathrm{e}^{\mathrm{i}l\alpha}\lambda(\alpha-2\pi m)][\mathrm{e}^{\mathrm{i}l\alpha'}\lambda(\alpha'-2\pi m)]^* = 2\pi\delta(\alpha-\alpha') \qquad (3.15)$$

or by the corresponding equations for $\mu(\beta)$. Equation (3.14) has the appearance of an orthonormality relation, while (3.15) looks like a completeness relation, which is, of course, not accidental and will be clarified in section 4.

For the particular choices of (3.7), we have

$$\text{(a)}\quad \lambda(\alpha) = \frac{2}{\alpha}\sin\frac{\alpha}{2} \equiv \operatorname{sinc}\left(\frac{\alpha}{2}\right),$$
$$\mu(\beta) = \left\{\begin{matrix} 1 & \text{for } -\pi < \beta < \pi \\ 0 & \text{else}\end{matrix}\right\} = \delta_{b0},$$
$$\text{(b)}\quad \lambda(\alpha) = \delta_{a0}, \qquad \mu(\beta) = \operatorname{sinc}\left(\frac{\beta}{2}\right), \qquad (3.16)$$
$$\text{(c)}\quad \lambda(\alpha) = \operatorname{sinc}\left(\frac{\alpha+2\pi a}{4}\right), \qquad \mu(\beta) = \operatorname{sinc}\left(\frac{\beta+2\pi b}{4}\right),$$
$$\text{where } a, b \text{ are the integers} \quad a = \left\lfloor\frac{\alpha}{2\pi}\right\rceil \quad \text{and} \quad b = \left\lfloor\frac{\beta}{2\pi}\right\rceil.$$

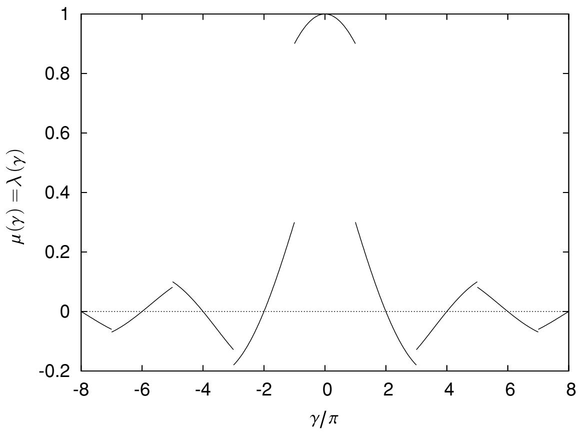

Figure 1. The discontinuous function $\lambda(\gamma) = \mu(\gamma)$ of choice (c) in equation (3.16) which is its own Fourier transform.

We emphasize the symmetry of choice (c), where $\lambda(\gamma) = \mu(\gamma)$ is its own Fourier transform. Figure 1 shows a plot of this function for $-8\pi < \gamma < 8\pi$.

4. Wavefunctions: discrete Zak basis

The wavefunctions for the discrete Zak basis states $|l, m\rangle$ are obtained from the wavefunctions in (3.3) by Fourier analysis. The position wavefunction

$$\begin{aligned}\langle x|l, m\rangle &= \int_{(2\pi)} \frac{\mathrm{d}\alpha}{2\pi} \int_{(2\pi)} \frac{\mathrm{d}\beta}{2\pi} \langle x|\alpha, \beta\rangle\, \mathrm{e}^{\mathrm{i}(l\alpha - m\beta)} \\ &= \frac{1}{\sqrt{x_0}}\, \mathrm{e}^{\mathrm{i}lp_0x/\hbar} \lambda(p_0x/\hbar - 2\pi m)\end{aligned} \tag{4.1}$$

reveals the physical significance of $\lambda(\alpha)$ of equation (3.11), and we recognize now that (3.14) and (3.15) are the position representations of the orthogonality and completeness relations in (2.13).

The corresponding momentum wavefunction

$$\langle p|l, m\rangle = \frac{1}{\sqrt{p_0}}\, \mathrm{e}^{-\mathrm{i}mx_0p/\hbar} \mu(x_0p/\hbar - 2\pi l) \tag{4.2}$$

involves $\mu(\beta)$, so that the Fourier relations of (3.13) are actually those between the position and momentum representations. We note the consistency with (2.18), which is quite explicitly visible in

$$\begin{aligned}&|l, m\rangle = U^{-l}V^{-m}|0, 0\rangle : \langle x|l, m\rangle = \mathrm{e}^{\mathrm{i}lp_0x/\hbar}\langle x - mx_0|0, 0\rangle, \\ &\langle p|l, m\rangle = \mathrm{e}^{-\mathrm{i}mx_0p/\hbar}\langle p - lp_0|0, 0\rangle,\end{aligned} \tag{4.3}$$

with

$$\langle x|0, 0\rangle = \lambda(p_0x/\hbar)/\sqrt{x_0}, \qquad \langle p|0, 0\rangle = \mu(x_0p/\hbar)/\sqrt{p_0}. \tag{4.4}$$

For a visualization of the states of the discrete Zak basis, it may be helpful to plot their Wigner functions [15][3], given by

$$W_{lm}(x, p) = W_{00}(x - mx_0, p - lp_0),$$
$$W_{00}(x, p) = 2\int \frac{\mathrm{d}\alpha}{2\pi}\lambda(p_0x/\hbar + \alpha)^* \,\mathrm{e}^{\mathrm{i}2\alpha p/p_0}\lambda(p_0x/\hbar - \alpha)$$
$$= 2\int \frac{\mathrm{d}\beta}{2\pi}\mu(x_0p/\hbar - \beta)^* \,\mathrm{e}^{\mathrm{i}2\beta x/x_0}\mu(x_0p/\hbar + \beta), \tag{4.5}$$

so that W_{lm} is centred at $(x, p) = (mx_0, lp_0)$ if W_{00} is centred at $(x, p) = (0, 0)$, as is the case for the three choices in (3.7) or (3.16). Choice (b) yields

$$W_{00}^{(\mathrm{b})}(x, p) = \begin{cases} 2\,\mathrm{snc}(1 - 2|x/x_0|, 2\pi p/p_0) & \text{for} \quad |x| < x_0/2, \\ 0 & \text{for} \quad |x| > x_0/2, \end{cases} \tag{4.6}$$

where

$$\mathrm{snc}(z, \alpha) = \frac{\sin(z\alpha)}{\alpha} = \frac{1}{2}\int_{-z}^{z} \mathrm{d}\zeta\, \mathrm{e}^{\mathrm{i}\zeta\alpha} \qquad \text{for} \quad 0 \leqslant z \leqslant 1 \tag{4.7}$$

is the incomplete sinc function, $\mathrm{sinc}(\alpha) = \mathrm{snc}(1, \alpha)$. For choice (a), the roles of x and p are interchanged, i.e.,

$$W_{00}^{(\mathrm{a})}(x, p) = W_{00}^{(\mathrm{b})}(x_0p/p_0, p_0x/x_0). \tag{4.8}$$

Figure 2 shows $W_{00}^{(\mathrm{b})}(x, p)$ for $-0.75 < x/x_0 < 0.75$ and $-5.25 < p/p_0 < 5.25$.

For choice (c), we obtain

$$W_{00}^{(\mathrm{c})}(x, p) = 2\sum_{j,k=0,1}(-1)^{(a+j)(b+k)}\,\mathrm{snc}\left(|1 - |t| - k|, (2a + s + j\,\mathrm{sgn}(s))\frac{\pi}{2}\right)$$
$$\times\,\mathrm{snc}\left(|1 - |s| - j|, (2b + t + k\,\mathrm{sgn}(t))\frac{\pi}{2}\right), \tag{4.9}$$

where $a = \lfloor 2x/x_0 \rceil$ and $b = \lfloor 2p/p_0 \rceil$ are the integers closest to $2x/x_0$ and $2p/p_0$, respectively, and s, t account for the differences,

$$2\frac{x}{x_0} = a + s, \qquad 2\frac{p}{p_0} = b + t \qquad \text{with} \quad -\frac{1}{2} \leqslant s, t \leqslant \frac{1}{2}. \tag{4.10}$$

The interchange $p_0x \leftrightarrow x_0p$ has no effect on $W_{00}^{(\mathrm{c})}$,

$$W_{00}^{(\mathrm{c})}(x, p) = W_{00}^{(\mathrm{c})}(x_0p/p_0, p_0x/x_0), \tag{4.11}$$

which is yet another manifestation of the symmetry that characterizes the phase convention (c) of (3.7).

In effect, then, the phase space is tiled by squares of size $\frac{1}{4}x_0p_0 = \frac{1}{2}\pi\hbar$, centred at $(x, p) = \left(\frac{1}{2}ax_0, \frac{1}{2}bp_0\right)$, with the pair s, t parameterizing the points of each square relative to the centre. At the centres of the tiles, we have

$$W_{00}^{(\mathrm{c})}\left(\frac{1}{2}ax_0, \frac{1}{2}bp_0\right) = 2\delta_{a0}\delta_{b0} = \begin{cases} 2 & \text{if } a = 0 \text{ and } b = 0, \\ 0 & \text{if } a \neq 0 \text{ or } b \neq 0. \end{cases} \tag{4.12}$$

Figure 3 shows $W_{00}^{(\mathrm{c})}(x, p)$ for $-2.25 < x/x_0, p/p_0 < 2.25$, the vicinity of the maximum at $(x, p) = (0, 0)$.

[3] They are normalized in accordance with $\int \frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar} W(x, p) = 1$ and bounded by $-2 \leqslant W(x, p) \leqslant 2$.

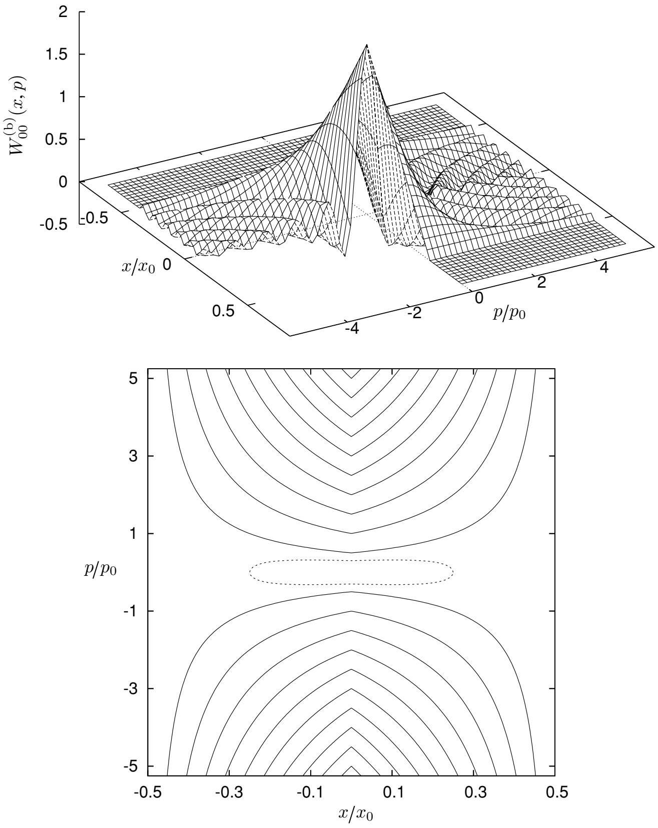

Figure 2. The Wigner function $W_{00}^{(b)}(x, p)$ corresponding to the ket $|0, 0\rangle$ for choice (b) in equation (3.16). Top: $W_{00}^{(b)}$ above the x, p plane, with the quadrant for $x > 0$, $p < 0$ cut out. Bottom: the $W_{00}^{(b)} = 0$ contour lines where the Wigner function changes sign (solid lines), and the $W_{00}^{(b)} = 1$ contour line where the Wigner function is at half-maximum value (dashed line).

These plots demonstrate that the Wigner functions in (4.6) and (4.9) are continuous in both x and p, although their marginals—of which

$$\begin{aligned}\int \mathrm{d}p\, W(x, p) &= p_0|\lambda(p_0x/\hbar)|^2,\\ \int \mathrm{d}x\, W(x, p) &= x_0|\mu(x_0p/\hbar)|^2\end{aligned}\tag{4.13}$$

are special cases—exhibit discontinuities as a rule. Indeed, it is a matter of inspection to verify that $W_{00}^{(b)}$ and $W_{00}^{(c)}$ are continuous.

5. Operator relations

We combine the wavefunctions of (4.1) and (4.2) with the Fourier series of (3.11) to establish

$$\begin{aligned}\frac{\langle x|\,\mathrm{e}^{\mathrm{i}\alpha L - \mathrm{i}\beta M}|p\rangle}{\langle x|p\rangle} &= \sqrt{x_0p_0}\,\mathrm{e}^{-\mathrm{i}xp/\hbar}\sum_{l,m}\langle x|l, m\rangle\,\mathrm{e}^{\mathrm{i}(l\alpha - m\beta)}\langle l, m|p\rangle\\ &= \chi\left(\frac{p_0x}{\hbar} + \alpha, \frac{x_0p}{\hbar}\right)^*\exp\left(\mathrm{i}\frac{\alpha x_0p - \beta p_0x}{4\pi\hbar}\right)\chi\left(\frac{p_0x}{\hbar}, \frac{x_0p}{\hbar} - \beta\right).\end{aligned}\tag{5.1}$$

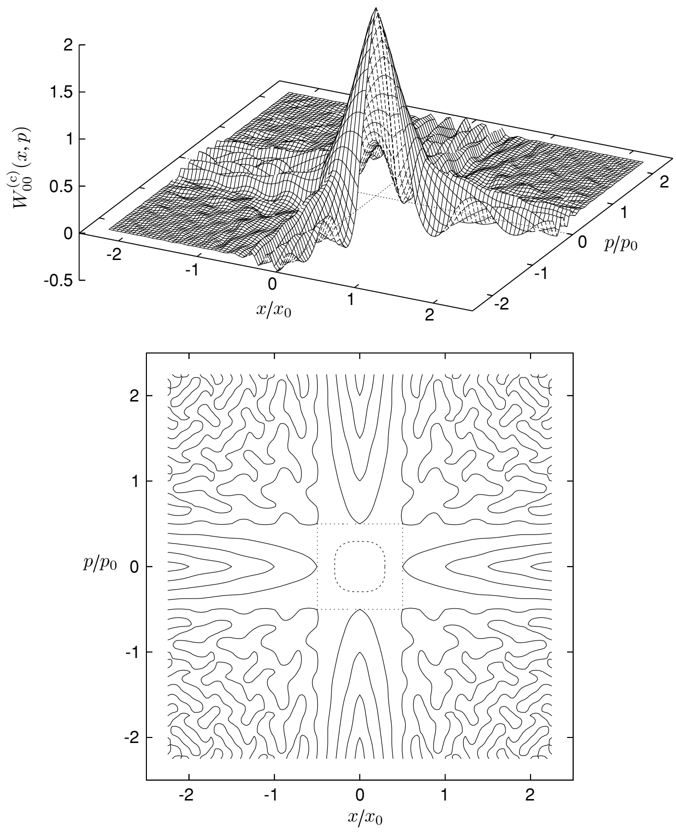

Figure 3. The Wigner function $W_{00}^{(c)}(x, p)$ corresponding to the ket $|0, 0\rangle$ for choice (c) in equation (3.16). Top: $W_{00}^{(c)}$ above the x, p plane, with the quadrant for $x > 0$, $p < 0$ cut out. Bottom: the $W_{00}^{(c)} = 0$ contour lines where the Wigner function changes sign (solid lines), and the $W_{00}^{(c)} = 1$ contour line where the Wigner function is at half-maximum value (dashed line). The dotted square identifies $-\frac{1}{2} < x/x_0, p/p_0 < \frac{1}{2}$, the central peak region of $W_{00}^{(c)} = 0$.

The normalized matrix elements of L and M are then obtained by differentiation,

$$\begin{aligned}\frac{\langle x|L|p\rangle}{\langle x|p\rangle} &= \frac{p}{2p_0} - \chi(\alpha,\beta)^*\frac{1}{\mathrm{i}}\frac{\partial}{\partial\alpha}\chi(\alpha,\beta)\bigg|_{\substack{\alpha = p_0x/\hbar \\ \beta = x_0p/\hbar}}, \\ \frac{\langle x|M|p\rangle}{\langle x|p\rangle} &= \frac{x}{2x_0} - \chi(\alpha,\beta)^*\mathrm{i}\frac{\partial}{\partial\beta}\chi(\alpha,\beta)\bigg|_{\substack{\alpha = p_0x/\hbar \\ \beta = x_0p/\hbar}}.\end{aligned} \tag{5.2}$$

The required derivatives are available in (3.10), and so we arrive at

$$\text{(a)}\quad L = \left\lfloor \frac{P}{p_0} \right\rceil, \qquad M = \frac{X}{x_0} - \delta^{(2\pi)}\left(\frac{x_0P}{\hbar} - \pi\right)\sin\frac{p_0X}{\hbar},$$

$$\text{(b)}\quad L = \frac{P}{p_0} - \delta^{(2\pi)}\left(\frac{p_0X}{\hbar} - \pi\right)\sin\frac{x_0P}{\hbar}, \qquad M = \left\lfloor \frac{X}{x_0} \right\rceil,$$

$$\text{(c)}\quad L = \frac{1}{2}\left(\frac{P}{p_0} + \left\lfloor\frac{P}{p_0}\right\rceil\right) - \delta^{(2\pi)}\left(\frac{p_0 X}{\hbar} - \pi\right)(-1)^{\lfloor P/p_0\rceil}\sin\frac{x_0 P}{2\hbar},$$
$$M = \frac{1}{2}\left(\frac{X}{x_0} + \left\lfloor\frac{X}{x_0}\right\rceil\right) - \delta^{(2\pi)}\left(\frac{x_0 P}{\hbar} - \pi\right)(-1)^{\lfloor X/x_0\rceil}\sin\frac{p_0 X}{2\hbar}, \tag{5.3}$$

which express L and M in terms of X and P for the three phase conventions of (3.7).

The reverse relations, which state the line variables X and P in terms of the torus variables U, L, and V, M, are at hand as soon as one observes that $L - P/p_0$ and $M - X/x_0$ are 2π periodic functions of $p_0 X/\hbar$ and $x_0 P/\hbar$, that is, they are functions of U and V. For the three choices of (3.7), we have

$$\frac{X}{x_0} = M + \begin{cases} \text{(a)} & \delta^{(2\pi)}(\beta - \pi)\sin\alpha, \\ \text{(b)} & \dfrac{\alpha}{2\pi} - \left\lfloor\dfrac{\alpha}{2\pi}\right\rceil, \\ \text{(c)} & \dfrac{1}{2}\left(\dfrac{\alpha}{2\pi} - \left\lfloor\dfrac{\alpha}{2\pi}\right\rceil\right) + \delta^{(2\pi)}(\beta - \pi)(-1)^{\lfloor\frac{\alpha}{2\pi}\rceil}\sin\dfrac{\alpha}{2}, \end{cases}$$
$$\frac{P}{p_0} = L + \begin{cases} \text{(a)} & \dfrac{\beta}{2\pi} - \left\lfloor\dfrac{\beta}{2\pi}\right\rceil, \\ \text{(b)} & \delta^{(2\pi)}(\alpha - \pi)\sin\beta, \\ \text{(c)} & \dfrac{1}{2}\left(\dfrac{\beta}{2\pi} - \left\lfloor\dfrac{\beta}{2\pi}\right\rceil\right) + \delta^{(2\pi)}(\alpha - \pi)(-1)^{\lfloor\frac{\beta}{2\pi}\rceil}\sin\dfrac{\beta}{2}, \end{cases} \tag{5.4}$$

where $e^{-i\alpha} = U$ and $e^{i\beta} = V$ are understood in the doubly periodic functions on the right-hand side, as illustrated by

$$(-1)^{\lfloor\frac{\alpha}{2\pi}\rceil}\sin\frac{\alpha}{2} = \sin\left(\frac{\alpha}{2} - \pi\left\lfloor\frac{\alpha}{2\pi}\right\rceil\right) = \sum_{j=-\infty}^{\infty}\frac{(-1)^{j+1}}{\left(j+\frac{1}{2}\right)\pi}\sin(j\alpha)$$
$$\longrightarrow \sum_{j=-\infty}^{\infty}\frac{(-1)^j}{i(2j+1)\pi}(U^j - U^{-j}). \tag{5.5}$$

For the example of choice (a), then, we have

$$\langle\alpha,\beta|X = x_0\left(-\frac{1}{i}\frac{\partial}{\partial\beta}\langle\alpha,\beta| + \sin\alpha\,\langle\alpha,\pi|\right),$$
$$\langle\alpha,\beta|P = p_0\left(\frac{1}{i}\frac{\partial}{\partial\alpha} + \frac{\beta}{2\pi} - \left\lfloor\frac{\beta}{2\pi}\right\rceil\right)\langle\alpha,\beta| \tag{5.6}$$

for the differential-operator representation of X and P in the continuous Zak representation.

6. Aharonov's modular operators

Adapted to the present notational conventions, Aharonov's modular position and momentum operators [17], X_m and P_m, are introduced by the symmetric relations

$$X = x_0 N_x + X_m, \qquad P = p_0 N_p + P_m, \tag{6.1}$$

where N_x and N_p are the integer operators

$$N_x = \lfloor X/x_0\rceil, \qquad N_p = \lfloor P/p_0\rceil. \tag{6.2}$$

In view of

$$U = \mathrm{e}^{-\mathrm{i}p_0X_\mathrm{m}/\hbar}, \qquad V = \mathrm{e}^{\mathrm{i}x_0P_\mathrm{m}/\hbar}, \tag{6.3}$$

we can regard the modular operators as the logarithms of the unitary operators U and V in the sense of

$$\begin{aligned} X_\mathrm{m} &= X - x_0\lfloor X/x_0\rceil = \frac{\mathrm{i}\hbar}{p_0}\ln\frac{\mathrm{e}^\epsilon + U}{\mathrm{e}^\epsilon + U^{-1}}\bigg|_{0>\epsilon\to 0}, \\ P_\mathrm{m} &= P - p_0\lfloor P/p_0\rceil = \frac{\hbar}{\mathrm{i}p_0}\ln\frac{\mathrm{e}^\epsilon + V}{\mathrm{e}^\epsilon + V^{-1}}\bigg|_{0>\epsilon\to 0}, \end{aligned} \tag{6.4}$$

where the limiting procedure ensures $-\frac{1}{2}x_0 < X_\mathrm{m} < \frac{1}{2}x_0$ and $-\frac{1}{2}p_0 < P_\mathrm{m} < \frac{1}{2}p_0$.

Therefore, (X_m, N_p) is a pair of complementary observables that is equivalent to (U, L); and, likewise, the pair (P_m, N_x) is as good as (V, M). But the two pairs (X_m, N_p) and (P_m, N_x) together do not refer to the two *independent* periodic variables of the torus because N_p does not commute with N_x,

$$\mathrm{i}[N_x, N_p] = \frac{1}{2\pi} - \delta^{(2\pi)}(p_0X/\hbar - \pi) - \delta^{(2\pi)}(x_0P/\hbar - \pi). \tag{6.5}$$

This non-commutativity is as it should be, because the pair (X_m, N_p) belongs to the phase convention (a) with $N_p = L^{(\mathrm{a})}$, whereas the pair (P_m, N_x) goes with phase convention (b), as $N_x = M^{(\mathrm{b})}$.

The non-commutativity of N_x and N_p reminds us that it is not possible to constrain a quantum system both to a finite x range and to a finite p range. It appears that, in the present context of the Zak bases, the symmetric compromise of imperfect simultaneous localization is given by the discrete Zak basis states of the symmetric phase convention (c). Indeed, $W_{00}^{(\mathrm{c})}(x, p)$ is very strongly peaked within the central square $-\frac{1}{2} \lesssim x/x_0, p/p_0 \lesssim \frac{1}{2}$ that is indicated in figure 3.

7. Hinting at an application: toroidal qubits

The three operators that are introduced by

$$\sigma_1 + \mathrm{i}\sigma_2 = [1 + (-1)^L]U, \qquad \sigma_3 = (-1)^L \tag{7.1}$$

are such that

$$\begin{aligned} &\mathrm{tr}\{\sigma_1\} = \mathrm{tr}\{\sigma_2\} = \mathrm{tr}\{\sigma_3\} = 0, \\ &\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1, \\ &\sigma_1\sigma_2 = \mathrm{i}\sigma_3, \qquad \sigma_2\sigma_3 = \mathrm{i}\sigma_1, \qquad \sigma_3\sigma_1 = \mathrm{i}\sigma_2, \end{aligned} \tag{7.2}$$

so that they are the components of a genuine Pauli vector operator $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. We can, therefore, use them to associate a qubit with the U, L degree of freedom. Likewise,

$$\tau_1 + \mathrm{i}\tau_2 = [1 + (-1)^M]V, \qquad \tau_3 = (-1)^M \tag{7.3}$$

defines a second toroidal qubit for the V, M degree of freedom.

For any quantum state on the line, there is then a two-qubit state with the statistical operator

$$\rho = \tfrac{1}{4}(1 + \vec{\sigma}\cdot\langle\vec{\sigma}\rangle + \langle\vec{\tau}\rangle\cdot\vec{\tau} + \vec{\sigma}\cdot\langle\vec{\sigma}\,\vec{\tau}\rangle\cdot\vec{\tau}). \tag{7.4}$$

The two vectorial expectation values $\langle\vec{\sigma}\rangle$, $\langle\vec{\tau}\rangle$ together with the dyadic expectation value $\langle\vec{\sigma}\,\vec{\tau}\rangle$ constitute the 15 parameters that specify the two-qubit state. Once their values are known, one can, for instance, decide whether ρ is separable or not, and thus whether an entangled qubit pair is concealed in the given state on the line. This application and others are, however, beyond the scope of the present paper on the properties of the Zak bases.

8. Summary

The unitary operators for displacement in position and momentum commute with one another when the product of the elementary displacements forms the elementary area $h = 2\pi\hbar$ in phase space. The Zak basis is composed of the complete set of eigenstates of these commuting operators and is therefore specified by two phase parameters.

Hence, the Zak basis maps the Hilbert space on the line onto the Hilbert space on a torus. We showed that for this mapping to be one to one, periodicity of the Zak basis states in their phase parameters is required and this, in turn, imposed discontinuity in the phases. We used the freedom present for the definition of this phases' discontinuity to consider three choices which we deemed most natural: (i) one phase parameter continuous, say α, and the other, β, discontinuous; (ii) conversely, α discontinuous while β continuous; (iii) symmetric discontinuity in both phases.

The resultant periodic Zak basis may be represented by a double Fourier series. The Fourier coefficients constitute a doubly discrete representation basis that is mutually unbiased with the Zak basis. The states of this discrete representation are characterized by pairs of integer eigenvalues. The Hermitian operators, for which these states are eigenstates with these eigenvalues, generate the fundamental rotations of the torus. Thus, with each consistent phase mapping convention of the line onto the torus we have one pair of mutually unbiased Zak bases.

Further, we considered briefly the relation between the Zak bases and Aharonov's modular operators and interpreted the operators employed by the latter in terms of the former: both, per force, reflect the basic fact that it is not possible to constrain a quantum system to finite domains in *both* position and momentum.

As a possible application of our study of the Zak operators on the torus, we mentioned the possibility of associating with a single degree of freedom on the line a pair of potentially entangled qubits. This is achieved by a rather simple, and quite natural, construction of two sets of Pauli operators in terms of the basic unitary operators of the Zak bases.

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